Journal of Engineering Mathematics Vol. II, No. 4, October 1968.
Wolters-Noordhoff Publishing.
Printed in the Netherlands

CRACK PROBLEM IN TWO-DIMENSIONAL ELASTICITY THEORY

S. M. Sharfuddin

(Rajshahi Government College, Rajshahi, East Pakistan)

SUMMARY

In this paper the stresses and displacement round a crack under various conditions have been investigated. The appropriate complex potentials are derived for four types of loading. These are: constant normal pressure on both sides of the crack with zero shear on the crack with zero principal stress at infinity, shear round the crack with zero finite stress at infinity, simple tension at infinity at a given angle to x-axis, shear at infinity. The case when the crack is subject to shear which is the case for nucleation of slip in fundamental metallurgical problem has been studied in detail. The problem has been solved after converting it into a Hilbert's Problem. Curves showing ischromatic lines (lines of constant stress) are sketched in.

1. *Introduction.*

The fundamental concept of Griffith's [1, 2] theory of rapture is that the bounding surfaces of a solid possess a surface tension, just as those of a liquid do, and that, when a crack spreads, the decrease in the strain energy is balanced by an increase in the potential energy due to surface tension. The calculation of the effect of the pressure of a crack on the energy of an elastic body is based on Inglish's [4] solution of the two-dimensional equation of elastic equilibrium in the space bounded by two concentric ellipses, the crack then being taken to be an ellipse of zero minor axis. The nature of the co-ordinate system used by Griffith does not lend itself easily to computation. Westergaard [9] gave simple solution for the case when it is supposed that the body is deformed by the opening of a crack under the action of a uniform pressure. The disadvantage of Westergaard's stress function is that it refers only to the case when the crack is subjected to uniform pressure. Sneddon [7, 8] considered the case when the crack is subjected to variable internal pressure by making use of the theory of Fourier Transforms and dual integral equations. Sanders [11], Green [10] and Koiter [12, 13] have tackled some crack problems of uniform tension. In this paper most general crack problem has been investigated. The problem has been solved after converting it into a Hilbert's Problem [3].

2. *Fundamental Equations.*

Following Muskhelishvili [5] when the body forces are absent the stress components $\widetilde{xx}$, $\widetilde{yy}$, $\widetilde{xy}$ and the complex displacement $D = u + iv$ can be expressed in terms of two analytic functions $\Omega(z)$, $\omega(z)$ of a complex variable $z = x + iy$, by means of the equations

$$2(\widetilde{xx} + \widetilde{yy}) = \Omega'(z) + \bar{\Omega}'(\bar{z}), \tag{2.1}$$

$$2(\widetilde{xx} - \widetilde{yy} + 2i\widetilde{xy}) = -z\,\Omega''(z) - \bar{\omega}''(\bar{z}), \tag{2.2}$$

$$8\mu D = 8\mu(u + iv) = k\,\Omega(z) - z\,\bar{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z}), \tag{2.3}$$

where μ is the shear modulus, $k = 3 - 4\eta$, η is Poisson's ratio and the prime denotes differentiation with respect to z. From (2.1) and (2.2) by subtraction

$$4(\widetilde{yy} - i\,\widetilde{xy}) = \Omega'(z) + \bar{\Omega}'(\bar{z}) + z\,\bar{\Omega}''(\bar{z}) + \bar{\omega}''(\bar{z}). \tag{2.4}$$

If T_1 and T_2 be the principal stresses at infinity in lines making an angle α and $\alpha + \frac{\pi}{2}$ with x-axis and if the force resultant round the crack be $X+iY$ then for large values of $|z|$ the complex potentials have the forms

$$\left.\begin{aligned} \Omega'(z) &= T_1 + T_2 - \frac{2(x+iy)}{\pi(1+k)}\frac{1}{z} + \sum_{n=2}^{\infty} A_n z^{-n}, \\ \omega''(z) &= -2(T_1 - T_2)e^{-2i\alpha} + \frac{2k}{\pi}\frac{x-iy}{1+k}\frac{1}{z} + \sum_{n=2}^{\infty} B_n z^{-n}. \end{aligned}\right\} \tag{2.5}$$

Introducing a new function $\gamma(z)$ where

$$\omega'(z) = -z\,\Omega'(z) - \bar{\gamma}(z); \qquad \bar{\gamma}(z) = \overline{\gamma(\bar{z})}. \tag{2.6}$$

The equations (2.4) and (2.3) can be written as

$$4(\widetilde{yy} - i\,\widetilde{xy}) = \Omega'(z) - \gamma'(\bar{z}) + (z - \bar{z})\,\bar{\Omega}''(\bar{z}). \tag{2.7}$$

$$8\mu D = k\,\Omega(z) + \gamma(\bar{z}) + (\bar{z} - z)\,\bar{\Omega}'(\bar{z}). \tag{2.8}$$

From (2.5) and (2.6) we get

$$\bar{\gamma}'(z) = -(T_1 + T_2) + 2(T_1 - T_2)e^{-2i\alpha} - \frac{2k}{\pi}\frac{x-iy}{1+k}\frac{1}{z} + O(z^{-2}). \tag{2.9}$$

3. *Boundary conditions on the surface of the crack.*

It will be assumed that the crack occupies the region $|x| < b$ of the real axis and that

$$\lim_{y \to 0} y\,\Omega''(z) = 0. \tag{3.1}$$

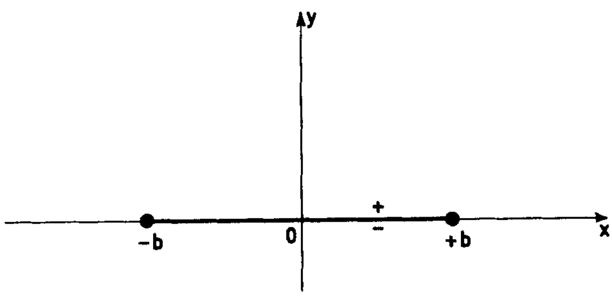

Fig 1.

From (2.7) and (3.1) the boundary conditions at the crack are

$$4(\widetilde{yy} - i\,\widetilde{xy})_+ = \Omega'_+(t) - \gamma'_-(t), \tag{3.2}$$

$$4(\widetilde{yy} - i\,\widetilde{xy})_- = \Omega'_-(t) - \gamma'_+(t), \tag{3.3}$$

where t is the complex number (actually real here) denoting points on the x-axis in the region $|x| < t$ and + and - sign indicate the surface values at the upper and lower surfaces of the crack respectively. By addition and subtraction we get

$$\begin{aligned}[\Omega'(t) - \gamma'(t)]_+ + [\Omega'(t) - \gamma(t)]_- \\ = 4(\widehat{yy}_+ + \widehat{yy}_-) - i(\widehat{xy}_+ + \widehat{xy}_-) \equiv q_1(t) \quad \text{(say)},\end{aligned} \tag{3.4a}$$

$$\begin{aligned}[\Omega'(t) + \gamma'(t)]_+ - [\Omega'(t) + \gamma'(t)]_- \\ = 4(\widehat{yy}_+ - \widehat{yy}_-) - i(\widehat{xy}_+ - \widehat{xy}_-) \equiv q_2(t) \quad \text{(say)}.\end{aligned} \tag{3.4b}$$

The solution of (3.4b) is given by Cauchy Integral

$$\Omega'(z) + \gamma'(z) = \frac{1}{2\pi i}\int \frac{q_2(t)}{t - z}dt + 2(T_1 - T_2)e^{2i\alpha}. \tag{3.5}$$

The equation (3.4a) constitutes particular case of Hilbert's Problem [3]. The required solution is given by

$$\Omega'(z) - \gamma'(z) = \frac{\chi(z)}{2\pi i}\int_L \frac{q_1(t)\,dt}{\chi_+(t)\ (t - z)} + \chi(z)P(z)\ , \tag{3.6}$$

where the Plemelj function [6] $\chi(z)$ in this case is given by

$$\chi(z) = (z^2 - b^2)^{-\frac{1}{2}}, \tag{3.7}$$

$\chi_+(t)$ denotes the limiting values on the line L (given by $|x| < b$, $y = 0$) when approached from the positive region $y > 0$ and $P(z)$ is a polynomial. Since $\chi(z) = 0(z^{-1})$ at infinity, the limiting conditions in (2.5) and (2.9) show that $P(z)$ is linear in z i.e.

$$P(z) = a_o + a_1 z\ . \tag{3.8}$$

4. *Evaluation of Integrals.*

It has been seen that (Muskhelishvili [5])

$$\int_L \frac{q_1(t)\ dt}{x_+(t)\ (t - z)} = -\frac{1}{2}\int_c \frac{q_1(t')\ dt}{x(t')\ (t' - z)} \tag{4.1}$$

where C is the contour as shown in Figure 2, the point z will be assumed outside the contour, $q_1(t')$ is obtained by replacing t by t' in $q_1(t)$.

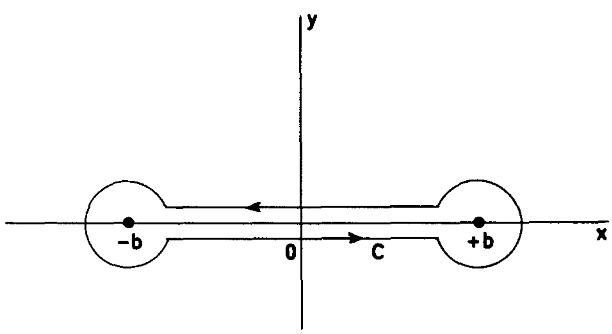

Fig 2.

Now for large values of t'

$$\frac{1}{x(t')} = (t'^2 - b^2) = t' + O(t'^{-1}) . \tag{4.2}$$

Taking $q_1(t) = C_1$ and $q_2(t) = C_2$ where C_1 and C_2 are constants

$$\frac{1}{\pi i}\int_L \frac{c_1 dt}{x_+(t)\ (t-z)} = C_1\left[(z^2-b^2)^{\frac{1}{2}} - z\right] . \tag{4.3}$$

Also

$$\frac{1}{2\pi i}\int_L \frac{q_2(t)}{t-z}dt = \frac{C_2}{2\pi i}\int_{-b}^{b}\frac{dt}{t-z} = \frac{C_2}{2\pi i}\log\frac{z-b}{z+b} . \tag{4.4}$$

Therefore we get

$$\begin{aligned}\Omega'(z) &= \frac{C_2}{4\pi i}\log\frac{z-b}{z+b} + \frac{C_1}{4}\left[1 - \frac{z}{(z^2-b^2)^{\frac{1}{2}}}\right] \\ &+ \frac{a_o + a_1 z}{2(z^2-b^2)} + (T_1 - T_2)e^{2i\alpha},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\gamma'(z) &= \frac{C_2}{4\pi i}\log\frac{z-b}{z+b} - \frac{C_1}{4}\left[1 - \frac{z}{(z^2-b^2)^{\frac{1}{2}}}\right] \\ &- \frac{a_o + a_1 z}{2(z^2-b^2)^{1/2}} + (T_1 - T_2)e^{2i\alpha}.\end{aligned}$$

for large values of $|z|$,

$$\Omega'(z) = -\frac{2bC_2}{4\pi iz} + \frac{a_o + a_1 z}{2z} + (T_1 - T_2)e^{2i\alpha} + O(z^{-2}), \tag{4.7}$$

$$\gamma'(z) = -\frac{2bC_2}{4\pi iz} - \frac{a_o + a_1 z}{2z} + (T_1 - T_2)e^{2i\alpha} + O(z^{-2}). \tag{4.8}$$

Comparing (4.7), (4.8) with (2.5) and (2.9) we get

$$\left.\begin{aligned} &C_2 = \frac{2i}{b}(x + iY),\ a_o = \frac{2\ (k-1)}{\pi\ (k+1)}(x + iY) . \\ &a_1 = 2(T_1 + T_2) - (T_1 - T_2)e^{2i\alpha}.\end{aligned}\right\} \tag{4.9}$$

If at any point the principal stresses are τ_1 and τ_2 then

$$\Theta = \tau_1 + \tau_2 \text{ and } |\phi| = \sqrt{\phi\bar{\phi}} = |\tau_1 - \tau_2| . \tag{4.10}$$

The lines of maximum shear will be at $\frac{\pi}{4}$ to the lines of principal stresses and the maximum shear is $\frac{1}{2}|\tau_1 - \tau_2|$.

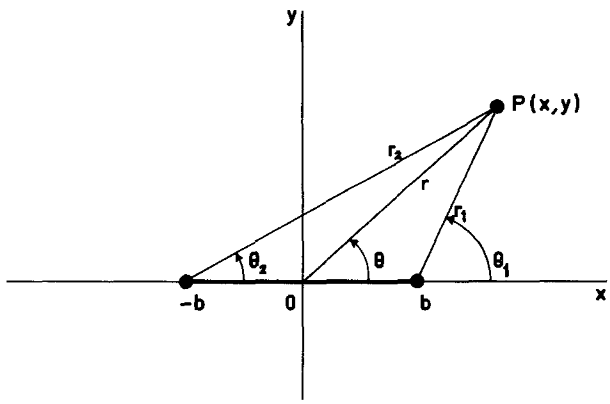

Fig 3.

5. *Special Cases.*

Case (i). We shall now consider the case when the crack is subject to the following conditions:

$$\left.\begin{array}{l} \text{Zero principal stresses at infinity, } T_1 = T_2 = 0. \\ \text{Zero shear on the crack, } \widetilde{xy}_+ = \widetilde{xy}_- = 0. \\ \text{Constant normal pressure p on both sides of the crack} \\ \qquad \widetilde{yy}_+ = \widetilde{yy}_- = -p). \end{array}\right\} \tag{5.1}$$

In this case

$$C_1 = -8p, \; C_2 = 0, \; a_o = 0, \; a_1 = 0. \tag{5.2}$$

Thus

$$\Omega'(z) = -2p\left[1 - \frac{z}{\sqrt{z^2 - b^2}}\right] = -\gamma'(z) \tag{5.3}$$

Therefore

$$\begin{aligned} \tau_1 + \tau_2 &= \Omega'(z) + \bar{\Omega}'(\bar{z}) \\ &= 2p\left[\frac{r}{\sqrt{r_1 r_2}} \cos\left(\frac{\theta_1 + \theta_2}{2}\right) - 1\right] \end{aligned} \tag{5.4}$$

Also

$$2\Phi = (\bar{z} - z)\,\bar{\Omega}'(\bar{z}) + \gamma'(\bar{z}) + \bar{\Omega}'(\bar{z}) \tag{5.5}$$

giving

$$(T_1 - T_2)^2 = \Phi\bar{\Phi} = 4p^2 b^4 y^2/(r_1 r_2)^3. \tag{5.6}$$

$$\text{Therefore maximum shear} = \tfrac{1}{2}\left|\tau_1 - \tau_2\right| = p\,\frac{r\sin\theta}{b}\left(\frac{b^2}{r_1 r_2}\right)^{3/2}. \tag{5.7}$$

The displacement D is given by

$$8\mu D = k\,\Omega(z) + \gamma(\bar{z}) + (\bar{z} - z)\,\bar{\Omega}'(\bar{z})$$
$$= -2kp\left[z - \sqrt{z^2 - b^2}\right] + 2p\left[\bar{z} - \sqrt{\bar{z}^2 - b^2}\right] + 2p\,(z - \bar{z})\left[1 - \frac{\bar{z}}{\sqrt{z^2 - b^2}}\right]. \quad (5.8)$$

When $y = 0$ i.e. $z = \bar{z}$,

$$8\mu D_{y=0} = 2p(1-k)\,x + 2p(k-1)\sqrt{r_1 r_2}\cos\frac{\theta_1 + \theta_2}{2} + i\,2p(k+1)\sqrt{r_1 r_2}\sin\frac{\theta_1 + \theta_2}{2}. \quad (5.9)$$

On the surface of the crack i.e. when $|x| < b$ and $\theta_1 = \pi$. $\theta_2 = 0$,

$$8\mu u_{y=0} = 2p(1-k)\,x, \quad 8\mu v_{y=0} = 2p(k+1)\sqrt{r_1 r_2}. \quad (5.10)$$

Thus we see that the tangential displacement of a point varies as its distance from the y-axis and vanishes at the origin. The transverse displacement is zero when $x = \pm b$ and finite ($\neq 0$) on the remaining part of the crack. Both the displacements are bounded.

Case (ii). Let us consider the case when the crack is subjected to shear and there is zero resultant stress at infinity.
In this case we have

$$\widehat{xy}_{+} = \widehat{xy}_{-} = -s \text{ (constant)}. \quad (5.11)$$

Thus $x + iY = 0$, $C_2 = 0 = a_o = a_1$, $C_1 = 8is$. (5.12)

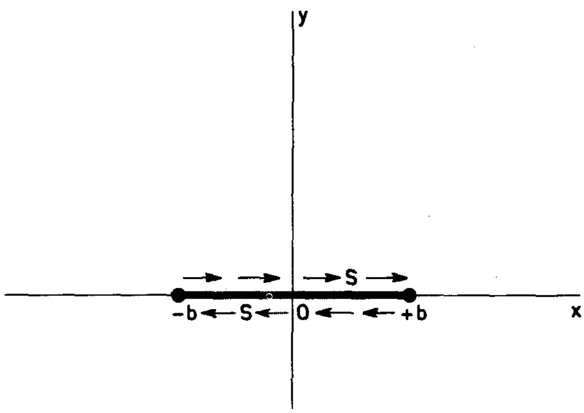

Fig 4.

Therefore in this case

$$\Omega'(z) = -\gamma'(z) = 2is\left[1 - \frac{z}{\sqrt{z^2 - b^2}}\right] \quad (5.13a)$$

$$\Phi = -2iS + \frac{2iS}{(r_1 r_2)^{3/2}}\left[iyb^2 e^{i\frac{3}{2}(\theta_1+\theta_2)} + r r_1 r_2 e^{-i(\theta - \frac{\theta_1+\theta_2}{2})}\right] \qquad (5.13b)$$

So we have

$$\tau_1 + \tau_2 = \frac{2rS}{\sqrt{r_1 r_2}} \operatorname{Sin}\left(\theta - \frac{\theta_1+\theta_2}{2}\right), \qquad (5.14)$$

$$|\tau_1 - \tau_2| = |\Phi| = 2S\left[1 + \frac{y^2 b^4}{(r_1 r_2)} + \frac{r^2}{r_1 r_2} + 2\frac{yb^2}{(r_1 r_2)} \operatorname{Sin}\frac{3}{2}(\theta_1+\theta_2) - 2\frac{r}{\sqrt{r_1 r_2}}\operatorname{Cos}\left(\theta - \frac{\theta_1+\theta_2}{2}\right) - 2\frac{yb^2 r}{(r_1 r_2)^2}\operatorname{Sin}(\theta+\theta_1+\theta_2)\right]^{\frac{1}{2}} \qquad (5.15)$$

In order to get the approximate value of the maximum shear near the end (b, 0) of the crack the finite terms of equation (5.13b) are dropped and we take

$$\theta = \theta_2 \underline{\Omega}\, 0, \quad y \underline{\Omega}\, r_1 \operatorname{Sin}\theta_1, \quad r \underline{\Omega}\, b, \quad r_1 \underline{\Omega}\, 2b. \qquad (5.16)$$

Thus

$$\Phi = \frac{iS}{2\sqrt{2}}\sqrt{\frac{b}{r_1}}\left[e^{i\frac{5}{2}\theta_1} + 3e^{i\frac{3}{2}\theta_1}\right]. \qquad (5.17)$$

Therefore near the end (b, 0)

$$|\Phi| = |\tau_1 - \tau_2| = \frac{S}{2}\sqrt{\frac{b}{r_1}}\,(5 + 3\operatorname{Cos} 2\theta_1)^{1/2}. \qquad (5.18)$$

and the maximum shear in this case $= \frac{1}{2}|\tau_1 - \tau_2|_{max.} = S\sqrt{\frac{b}{2r_1}}$. (5.19)

Also

$$8\mu D = 2iSK\left[re^{i\theta} - \sqrt{r_1 r_2}\, e^{i\frac{\theta_1+\theta_2}{2}}\right] - 2iS\left[re^{-i\theta} - \sqrt{r_1 r_2}\, e^{-i\frac{\theta_1+\theta_2}{2}}\right] - 4Sy\left[1 - \frac{r}{\sqrt{r_1 r_2}}\, e^{-i\left(\theta - \frac{\theta_1+\theta_2}{2}\right)}\right]. \qquad (5.20)$$

Since we may write $y = r_1 \operatorname{Sin}\theta_1 = r_2 \operatorname{Sin}\theta_2$, $0 < \theta < \pi$, we note that the displacements are finite everywhere. Now consider $|x| < b$.
On $y = 0_+$ and $y = 0_-$ i.e. on the upper and lower faces of the crack $\theta_1 = \pi$, $\theta_2 = 0$. Thus

$$8\mu D_{y=0} = 2iSr\left[ke^{i\theta} - e^{-i\theta}\right] + 2S\sqrt{r_1 r_2}\,(k+1). \qquad (5.21)$$

Therefore

$$8\mu u_{y=0} = 2S\sqrt{r_1 r_2}\,(k+1), \quad (\theta = 0),$$

$$8\mu v_{y=0} = 2S(k-1)x, \quad (\theta = 0), \qquad (5.22)$$

and

$$\left.\begin{aligned} 8\mu u_{y=0} &= -2S\sqrt{r_1 r_2}(k+1), \quad (\theta = \pi), \\ 8\mu v_{y=0} &= -2S(k-1)x \quad, \quad (\theta = \pi). \end{aligned}\right\} \tag{5.23}$$

Thus in this case both the tangential and transverse displacements are bounded. The transverse displacement vanishes at the origin and varies with the x-co-ordinate of the point.

Case (iii). Let us now consider the case when the crack is stress-free and $T_2 = 0$ i.e. simple tension at infinity at an angle α to x-axis. In this case

$$C_1 = C_2 = a_0 = 0, \quad a_1 = 2T_1(1 - e^{2i\alpha}). \tag{5.24}$$

and therefore

$$\left.\begin{aligned} \Omega'(z) &= \frac{T_1 z(1 - e^{2i\alpha})}{(z^2 - b^2)^{\frac{1}{2}}} + T_1 e^{2i\alpha}, \\ \gamma'(z) &= -\frac{T_1 z(1 - e^{2i\alpha})}{(z^2 - b^2)^{\frac{1}{2}}} + T_1 e^{2i\alpha}. \end{aligned}\right\} \tag{5.25}$$

Thus

$$\tau_1 + \tau_2 = \frac{2T_1 r}{\sqrt{r_1 r_2}} \operatorname{Cos}\left(\theta - \frac{\theta_1 + \theta_2}{2}\right) - \frac{2T_1 r}{\sqrt{r_1 r_2}} \operatorname{Cos}\left(\theta - \frac{\theta_1 + \theta_2}{2} + 2\alpha\right) + $$
$$+ 2T_1 \operatorname{Cos} 2\alpha, \tag{5.26}$$

$$\Phi = \frac{2iyb^2 T_1}{(r_1 r_2)^{\frac{3}{2}}} e^{i\frac{3}{2}(\theta_1 + \theta_2)} + \frac{iT_1}{\sqrt{r_1 r_2}} e^{-i\left(\theta - \frac{\theta_1+\theta_2}{2}\right)} \operatorname{Sin} 2\alpha + T_1 \operatorname{Cos} 2\alpha. \tag{5.27}$$

When $\alpha = \pi/2$

$$\tau_1 + \tau_2 = 4\frac{T_1 r}{\sqrt{r_1 r_2}} \operatorname{Cos}\left(\theta - \frac{\theta_1 + \theta_2}{2}\right) - 2T_1 \tag{5.28}$$

$$|\tau_1 - \tau_2| = |\Phi| = T_1\left[\frac{4y^2 b^4}{(r_1 r_2)} + 4\frac{yb^2}{(r_1 r_2)^{3/2}} \operatorname{Sin}\frac{3}{2}(\theta_1 + \theta_2) + 1\right]^{\frac{1}{2}} \tag{5.29}$$

To obtain the approximate value of the maximum shear near the end (b, 0) of the crack, we take

$$\Phi = 2i\frac{yb^2 T_1}{(r_1 r_2)^{3/2}} e^{i\frac{3}{2}(\theta_1 + \theta_2)}. \tag{5.30}$$

giving

$$|\tau_1 - \tau_2| = |\Phi| = 2\,T_1 \frac{yb^2}{(r_1 r_2)^{3/2}}\,. \tag{5.31}$$

Thus the maximum shear is $(yb^2 T_1)/(r_1 r_2)^{3/2}$.

The displacement is given by

$$8\mu D = T_1 r(Ke^{i\theta} + e^{-i\theta})e^{2i\alpha} + T_1\sqrt{r_1 r_2}(1-e^{2i\alpha})\left\{Ke^{i\frac{\theta_1+\theta_2}{2}} - e^{-i\frac{\theta_1+\theta_2}{2}}\right\} + $$
$$- 2iyT_1\left\{-1 + \frac{2r}{\sqrt{r_1 r_2}} e^{-i(\theta - \frac{\theta_1+\theta_2}{2})}\right\}. \tag{5.32}$$

If $\alpha = \pi/2$,

$$8\mu D = -T_1 r(Ke^{i\theta} + e^{-i\theta}) + 2T_1\sqrt{r_1 r_2}\left\{Ke^{i\frac{\theta_1+\theta_2}{2}} - e^{-i\frac{\theta_1+\theta_2}{2}}\right\} + $$
$$- 2iyT_1\left\{-1 + \frac{2r}{\sqrt{r_1 r_2}} e^{-i\frac{\theta_1+\theta_2}{2}}\right\} \tag{5.33}$$

When $|x| < b$, $y = 0_+$ and $y = 0_-$, i.e. on the upper and lower faces of the crack $\theta_1 = \pi$, $\theta_2 = 0$ and

$$8\mu D_{y=0} = -T_1 r(ke^{i\theta} + e^{-i\theta}) + 2i\,T_1\sqrt{r_1 r_2}(k+1), \tag{5.34}$$

Thus

$$\left.\begin{aligned} 8\mu u_{y=0} &= -T_1(k+1)\,x, && (\theta = 0), \\ 8\mu v_{y=0} &= 2T_1(k+1)\sqrt{r_1 r_2}, && (\theta = 0), \end{aligned}\right\} \tag{5.35}$$

$$\left.\begin{aligned} 8\mu u_{y=0} &= T_1(k+1)\,x, && (\theta = \pi), \\ 8\mu v_{y=0} &= 2T_1(k+1)\sqrt{r_1 r_2}, && (\theta = \pi). \end{aligned}\right\} \tag{5.36}$$

Thus in this case the tangential displacement vanishes with the x-co-ordinate of the point while both the transverse and tangential displacements remain bounded.

Case (iv). Finally we shall consider the case when the crack is subject to the following conditions:

Shear at infinity, i.e. $T_1 = -T_2$, $\alpha = \frac{\pi}{4}$, this gives shear T_1 parallel to the axes. In this case

$$C_1 = C_2 = 0,\ a_o = 0,\ a_1 = -4T_1 e^{2i\alpha} = -4iT_1. \tag{5.37}$$

So we get

$$\Omega'(z) = 2i\,T_1\left[1 - \frac{z}{\sqrt{z^2 - b^2}}\right],\quad \gamma'(z) = 2i\,T_1\left[\frac{z}{\sqrt{z^2 - b^2}}\right]. \tag{5.38}$$

$$\Phi = -2\,\frac{y\,b^2}{(r_1 r_2)^{3/2}}\,T_1 e^{i\frac{3}{2}(\theta_1+\theta_2)} + + T_1\left[2i + \frac{2r}{\sqrt{r_1 r_2}}\,T\ \mathrm{Sin}\left(\theta - \frac{\theta_1+\theta_2}{2}\right)\right]. \tag{5.39}$$

Therefore

$$\tau_1 + \tau_2 = \frac{2r}{\sqrt{r_1 r_2}}\,T_1\,\mathrm{Sin}\left(\theta - \frac{\theta_1+\theta_2}{2}\right). \tag{5.40}$$

$$|\tau_1 - \tau_2| = |\phi| = 2T_1\left[1 + \frac{y^2 b^4}{(r_1 r_2)^3} - 2\,\frac{y b^2}{(r_1 r_2)^{3/2}}\mathrm{Sin}\,\frac{3}{2}(\theta_1+\theta_2) + \frac{r^2}{r_1 r_2}\mathrm{Sin}^2\left(\theta - \frac{\theta_1+\theta_2}{2}\right) - \frac{2 y b^2 r}{(r_1 r_2)}\mathrm{Sin}\left(\theta - \frac{\theta_1+\theta_2}{2}\right)\mathrm{Cos}\,\frac{3}{2}(\theta_1+\theta_2)\right] \tag{5.41}$$

To get the approximate value of the maximum shear near the end (b, 0) we take

$$\Phi = -2T_1\frac{b^2 y}{(r_1 r_2)^{3/2}}\left[\mathrm{Cos}\,\frac{3}{2}(\theta_1+\theta_2) + i\,\mathrm{Sin}\,\frac{3}{2}(\theta_1+\theta_2)\right] + + \frac{2r}{\sqrt{r_1 r_2}}\,T_1\,\mathrm{Sin}\left(\theta - \frac{\theta_1+\theta_2}{2}\right). \tag{5.42}$$

Therefore

$$|\phi|^2 \leqslant 4\,\frac{T_1^2}{r_1 r_2}\left(\frac{b^2 y}{r_1 r_2} + r\right)^2. \tag{5.43}$$

Thus the maximum shear lies between

$$\frac{T_1^2}{\sqrt{r_1 r_2}}\left(\frac{b^2 y}{r_1 r_2} \sim r\right) \text{ and } \frac{T}{\sqrt{r_1 r_2}}\left(\frac{b^2 y}{\sqrt{r_1 r_2}} + r\right)$$

$$8\mu D = 2irT_1(Ke^{i\theta} + Ke^{-i\theta}) - 2iT_1\sqrt{r_1 r_2}\left[K e^{i\frac{\theta_1+\theta_2}{2}} - e^{-i\frac{\theta_1+\theta_2}{2}}\right] - 4yT_1\left[1 - \frac{r}{\sqrt{r_1 r_2}}\,e^{-i\left(\theta - \frac{\theta_1+\theta_2}{2}\right)}\right]. \tag{5.44}$$

When $|x| < b$, on $y = 0_+$ and $y = 0_-$, i.e. on the upper and lower faces of the crack $\theta_1 = \pi$, $\theta_2 = 0$, and

$$8\mu D_{y=0} = 2irT_1(ke^{i\theta} + e^{-i0}) + 2T_1(k+1)\sqrt{r_1 r_2}. \tag{5.45}$$

Therefore

$$\left.\begin{aligned} 8\mu u_{y=0} &= 2T_1(k+1)\sqrt{r_1 r_2}, \quad (\theta = 0), \\ 8\mu v_{y=0} &= 2T_1(k+1)\,x, \quad (\theta = 0) \end{aligned}\right\} \tag{5.46}$$

and

$$\left.\begin{aligned} 8\mu u_{y=0} &= 2T_1(k+1)\sqrt{r_1 r_2}, \quad (\theta = \pi), \\ 8\mu v_{y=0} &= -2T_1(k+1)\,x, \quad (\theta = \pi). \end{aligned}\right\} \tag{5.47}$$

Thus in this case, like case (ii), the transverse displacement varies with the x-co-ordinate and vanishes at the origin while both the tangential and transverse displacements are bounded.

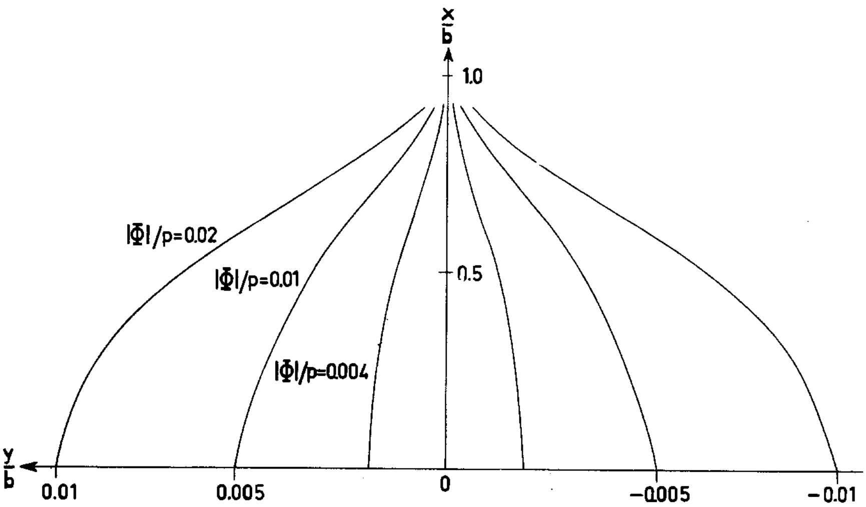

Fig 5.

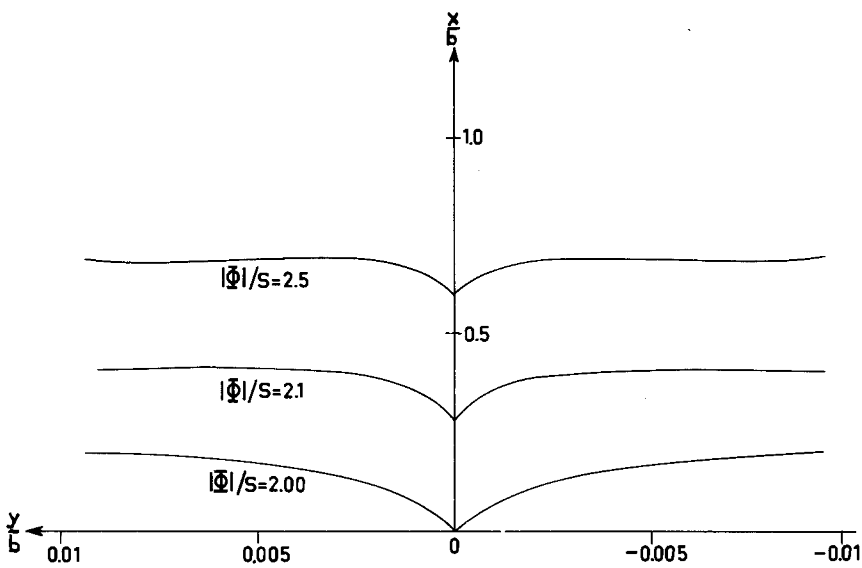

Fig 6.

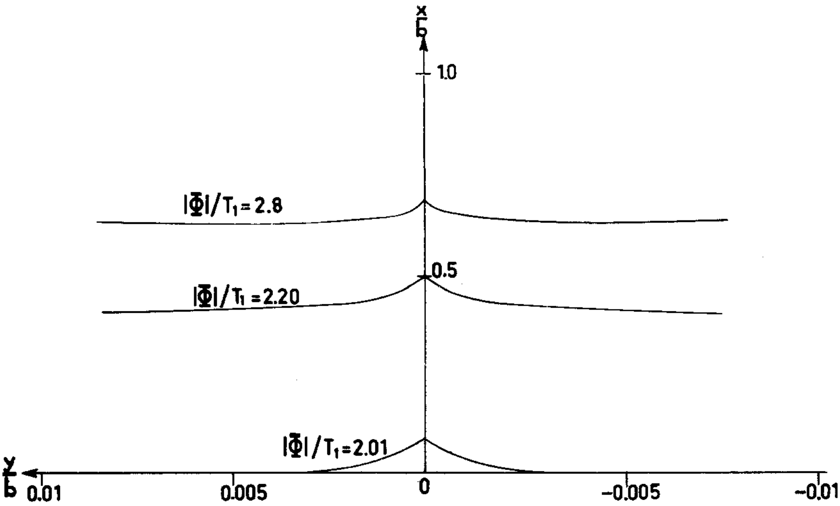

Fig 7.

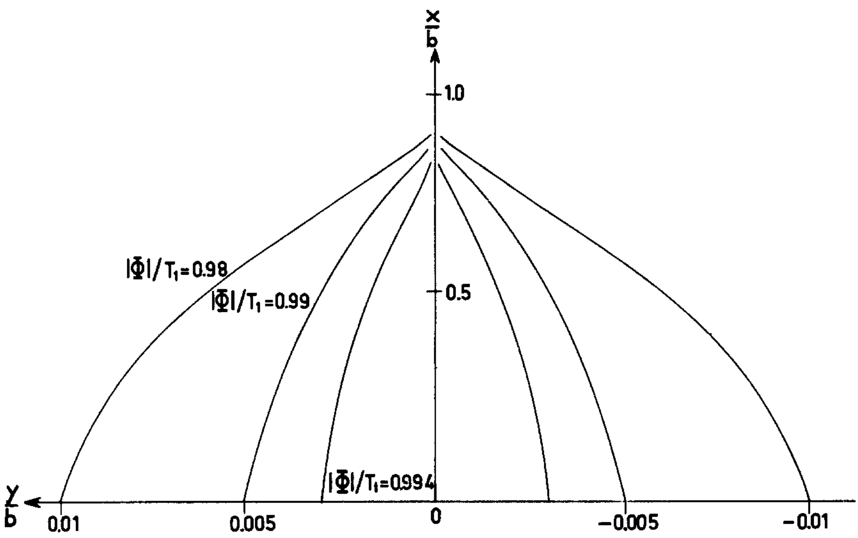

Fig 8.

6. *Acknowledgements.*

The author wishes to express his sincere gratitude to Mr Shamsul Huq SQA, Vice-Chancellor, Rajshahi University and to Mr M.A.Hye, Principal Rajshahi College for making this work possible.

He is also indebted to Mr M. Mujibul Huq CSP, Education Secretary, Government of East Pakistan and to Mr M. F. Khan TQA, DPI East Pakistan for their constant encouragement.

Acknowledgements are also due to the referees for their helpful comments and advice.

REFERENCES

1. Griffith, A. A: Phil. Trans. A. 221 (1921) 180
2. Griffith, A. A: Proc. Intern. Congress of Applied Math. (Delft). (1924) 55.
3. Hilbert, D: Internationalen Mathematiker Kongress Heidelberg (1904) 233.
4. Inglisch, C. E: Engineering 95 (1913) 415.
5. Muskhelishvili, N. I: Some Basic Problems of the Mathematical Theory of Elasticity, 4th. Edn. (Moscow-Leningrad 1954).
6. Plemelj, J: Monatshefte fur Math. u. Physik (1908) 205.
7. Sneddon, I. N: Proc. Roy. Soc. (London) A 187 (1946) 229.
8. Sneddon, I. N: Fourier Transforms (McGraw-Hill 1951).

Crack problem in two-dimensional elasticity theory.
9. Westergaard, H. W: Journal Applied Mech. 6 A (1939) 49.
10. Green, A. E & Sneddon, I. N: Cambrige Phil. Soc. 46 (1950) 159.
11. Sanders, J. L: Journal of Applied Mech, Vol. 33 (1966) 561.
12. Koiter, W. T: Ing-Arch. 28 (1959) 168.
13. Koiter, W. T: Problem of Continuum Mechanics, pp 246 - 259.

[Received February 2, 1968
and in revised form June 10, 1968]